

Boltzmann Equations from First Principles without Gradient Expansion

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Abstract

We show from first principles the emergence of classical Boltzmann equations for a generic quantum field that couples to a collection of background fields and sources. Our analysis is based on relativistic nonequilibrium quantum field theory as described by the Kadanoff-Baym equations and does not rely on the usual semi-classical approximations, *i.e.* on-shell condition, close-to-equilibrium assumption, gradient expansion and the Kadanoff-Baym ansatz. Instead, we find simple analytical solutions to the full Kadanoff-Baym equations by using the Wentzel-Kramers-Brillouin (WKB) method, and discuss under which circumstances Boltzmann behavior emerges.

I. INTRODUCTION

Nonequilibrium phenomena play a crucial role in many areas of physics, including the early history of the universe, heavy ion collisions, condensed matter physics and quantum information. In the era of precision cosmology and with the arrival of the LHC and RHIC experiments, in particular the first two applications, which require a relativistic description, have gained considerable interest. Transport in nonequilibrium situations can often in very good approximation be described by Boltzmann equations (BEs). These assume that the system can be characterized by a number of distribution functions for classical particles, which propagate freely between isolated interactions and carry no memory of their history. However, the definition of asymptotic states, on which the single particle description is based, is ambiguous in a dense plasma. What is more, the standard BEs by construction cannot describe memory and off-shell effects or quantum coherence. Usually these effects are treated by effective kinetic equations of the Boltzmann type [1–21], *i.e.* by a set of first order differential equations for generalized distribution functions that are local in time. As the above issues are conceptual, their range of validity and possible corrections cannot be determined within a framework of BEs and require a derivation from first principles.

The full equations of motion of nonequilibrium quantum field theory, on which first-principle derivations of the BEs usually are based, are known as Kadanoff-Baym equations (KBEs) [22]. These equations, being coupled second order integro-differential equations, are considerably more complicated than BEs. Analytical approaches to establish a connection between out-of-equilibrium quantum fields and kinetic equations usually make a number of assumptions on the KBEs *before* they are solved (*e.g.* Refs. [10, 11, 13, 16–21]): These include a gradient expansion and/or close-to-equilibrium assumption for all fields, the quasiparticle approximation and the Kadanoff-Baym ansatz. In this letter, we show how the *full* KBEs can be solved by using the WKB [23] method (for earlier uses of the WKB method in a similar context see *e.g.* Refs. [11, 24]). For the case of a single real scalar field, we find analytical expressions for its two-point correlation functions that are valid at all times, and discuss under which circumstances they exhibit Boltzmann behavior. The approach can be generalized to scenarios with multiple fields; backreaction can be included self-consistently.

II. NONEQUILIBRIUM QUANTUM FIELD THEORY

We consider the dynamics of a real scalar field ϕ that is described by relativistic quantum field theory. The field ϕ weakly couples to a background, possibly containing many degrees of freedom whose dynamics is in principle known and that we refer to as χ_i . The Lagrangian reads

$$\mathcal{L} = \frac{1}{2}\partial^\mu\phi\partial_\mu\phi - \frac{1}{2}m(t)^2\phi^2 - \phi \mathcal{O}[\chi_i] + \mathcal{L}_{\chi_i} , \quad (1)$$

where $\mathcal{O}[\chi_i]$ denotes a generic combination of fields χ_i , and \mathcal{L}_{χ_i} determines the dynamics of χ_i (we use $\hbar = c = 1$). We allow a time-dependent mass $m(t)$ to account for Hubble expansion when interpreting t as conformal time.

In quantum physics, any thermodynamic system can be characterized by a density matrix ϱ . Knowledge of the density matrix, or knowledge of all n -point functions $\langle\phi(x_1)\cdots\phi(x_n)\rangle$ of the fields, allows to compute expectation values for all observables at all times (with $\langle\cdots\rangle \equiv \text{tr}[\varrho\cdots]$). However, most quantities of practical interest for which one formulates a Boltzmann equation can be expressed in terms of one- and two-point functions; this includes the energy-momentum tensor and charge densities. It is, therefore, usually sufficient to track the time evolution of these.

An out-of-equilibrium quantum field has two independent connected two-point functions. In case of ϕ they are conveniently chosen as

$$\begin{aligned} \Delta^-(x_1, x_2) &\equiv i\langle[\phi(x_1), \phi(x_2)]\rangle , \\ \Delta^+(x_1, x_2) &\equiv \frac{1}{2}\langle\{\phi(x_1), \phi(x_2)\}\rangle , \end{aligned} \quad (2)$$

with the obvious symmetry relations $\Delta^\pm(x_2, x_1) = \pm\Delta^\pm(x_1, x_2)$. Here $[\cdot, \cdot]$ and $\{\cdot, \cdot\}$ are commutator and anti-commutator, respectively. $\Delta^-(x_1, x_2)$ is known as *spectral function* and basically encodes information about the spectrum of resonances. The *statistical propagator* $\Delta^+(x_1, x_2)$ carries information about the occupation numbers of different modes and can be viewed as a generalization of the classical particle distribution function. We have in mind applications in cosmology and restrict the analysis to spatially homogeneous systems. Then, the correlation functions only depend on relative spatial coordinates $\mathbf{x}_1 - \mathbf{x}_2$ etc., and it is convenient to perform a spatial Fourier transform in these coordinates, yielding functions like $\Delta_\mathbf{q}^\pm(t_1, t_2) \equiv \int d^3(\mathbf{x}_1 - \mathbf{x}_2) e^{-i\mathbf{q}(\mathbf{x}_1 - \mathbf{x}_2)} \Delta^\pm(x_1, x_2)$.

In a general out-of-equilibrium system the two-point functions $\Delta_{\mathbf{q}}^{\pm}(t_1, t_2)$ have to be found as solutions to the KBEs

$$(\partial_{t_1}^2 + \omega_{\mathbf{q}}(t_1)^2) \Delta_{\mathbf{q}}^{-}(t_1, t_2) = - \int_{t_2}^{t_1} dt' \Pi_{\mathbf{q}}^{-}(t_1, t') \Delta_{\mathbf{q}}^{-}(t', t_2) \quad (3)$$

$$(\partial_{t_1}^2 + \omega_{\mathbf{q}}(t_1)^2) \Delta_{\mathbf{q}}^{+}(t_1, t_2) = - \int_{t_i}^{t_1} dt' \Pi_{\mathbf{q}}^{-}(t_1, t') \Delta_{\mathbf{q}}^{+}(t', t_2) + \int_{t_i}^{t_2} dt' \Pi_{\mathbf{q}}^{+}(t_1, t') \Delta_{\mathbf{q}}^{-}(t', t_2) , \quad (4)$$

where $\omega_{\mathbf{q}}(t)^2 \equiv m(t)^2 + \mathbf{q}^2$ (note that in equilibrium Δ_q^{\pm} would only depend on $t_1 - t_2$), and t_i denotes the initial time of the system. The KBEs can be derived within the Schwinger-Keldysh formalism, see *e.g.* [25–29]; for simplicity we assumed Gaussian initial correlations for ϕ . The boundary conditions for $\Delta_{\mathbf{q}}^{-}$ are fixed by microcausality and canonical quantization for a real scalar field, $\Delta_{\mathbf{q}}^{-}|_{t_1=t_2} = 0$, $\partial_{t_1} \Delta_{\mathbf{q}}^{-}|_{t_1=t_2} = -\partial_{t_2} \Delta_{\mathbf{q}}^{-}|_{t_1=t_2} = 1$ and $\partial_{t_1} \partial_{t_2} \Delta_{\mathbf{q}}^{-}|_{t_1=t_2} = 0$. The boundary conditions for $\Delta_{\mathbf{q}}^{+}|_{t_1=t_2=t_i}$ are determined by the physical initial conditions of the system at time t_i . Below, we will drop momentum indices \mathbf{q} when possible.

The quantities Π^{\pm} appearing in (3) and (4) are the self-energies of ϕ ; in analogy to (2) they are at leading order in $\mathcal{O}[\chi_i]$ given by

$$\begin{aligned} \Pi^{-}(x_1, x_2) &= \langle [\mathcal{O}[\chi_i(x_1)], \mathcal{O}[\chi_i(x_2)]] \rangle , \\ \Pi^{+}(x_1, x_2) &= -\frac{i}{2} \langle \{ \mathcal{O}[\chi_i(x_1)], \mathcal{O}[\chi_i(x_2)] \} \rangle , \end{aligned} \quad (5)$$

and contain information about the interaction between ϕ and the background fields χ_i . They can be calculated in terms of the two-point functions of χ_i within the 2PI formalism (see *e.g.* Ref. [25, 26] for details).

III. TOWARDS BOLTZMANN EQUATIONS

We will discuss the emergence of a description of ϕ in terms of effective kinetic equations by using analytical solutions of the full KBEs that are found with the WKB method. To this end, we make the following assumptions (we also send $t_i \rightarrow -\infty$, effects of finite t_i are discussed below):

1) The self-energy $\Pi^{\pm}(t_1, t_2)$ is strongly damped with respect to the relative time $|t_1 - t_2|$, approaching zero for $|t_1 - t_2| \gtrsim \tau_{\text{int}}$, where we introduced the *interaction time* τ_{int} . Here, τ_{int} can be considered as definition for the duration of *e.g.* scattering events. It is then possible

to define one-sided Fourier transforms of the self-energies with respect to relative time,

$$\tilde{\Pi}^{\pm}(t, \omega) \equiv \int_0^{\infty} dz e^{i\omega z} \Pi^{\pm}(t, t-z) . \quad (6)$$

In equilibrium the minus-component would correspond to the common retarded self-energy, $\tilde{\Pi}^{-}(t, \omega) = \Pi^R(\omega)$.

2) We assume that for fixed time t the pole structure of $(\omega^2 - \omega_{\mathbf{q}}^2(t) - \tilde{\Pi}^{-}(t, \omega))^{-1}$ is dominated by the root $\omega = \hat{\Omega}_t \equiv \Omega_t - \frac{i}{2}\Gamma_t$, with

$$\Omega_t \equiv \sqrt{\omega_{\mathbf{q}}^2 + \text{Re}\tilde{\Pi}^{-}(t, \hat{\Omega}_t)} , \quad \Gamma_t \equiv -\frac{\text{Im}\tilde{\Pi}^{-}(t, \hat{\Omega}_t)}{\Omega_t} . \quad (7)$$

Here, Ω_t and Γ_t are the energy and damping rate of the ϕ -resonance, and we assume *weak damping* with $\Gamma_t \ll \Omega_t$. The generalization to the case with more roots (interpreted as collective ϕ excitations) is straightforward.

3) Three time scales of the system are relevant for our discussion: The above interaction time τ_{int} , the damping rate Γ_t of the field ϕ , and the characteristic rate $H \sim \dot{\Gamma}_t/\Gamma_t \sim \dot{\Omega}_t/\Omega_t$ with which the field ϕ changes its properties (*e.g.* due to a temperature change of the fields χ_i in an expanding universe; then H denotes the Hubble rate). The main assumption, underlying all derivations of kinetic equations, is the separation of time scales:

$$\Gamma_t \ll \tau_{\text{int}}^{-1} \quad \text{and} \quad \dot{\Gamma}_t/\Gamma_t, \dot{\Omega}_t/\Omega_t \ll \tau_{\text{int}}^{-1} . \quad (8)$$

The first condition implies that the duration of one individual collision is shorter than the average time between different collisions; the second condition implies that the field does not significantly change its properties during a single collision.

Localization of $\Pi^{\pm}(t_1, t_2)$ in time occurs for different physical reasons: If all fields that appear inside Π^{\pm} reside in a thermal bath, their propagators are exponentially damped with the thermal damping rate; in that case Π^{\pm} is at least damped with the same rate. More generally, when considering the scattering of particles, the duration of a scattering event typically is related to the de Broglie wavelengths of the interacting particles and is of the order Ω^{-1} ; formally the localization arises from the momentum integrals inside of loops. Finally, virtual particles with energies $\sim M$ can only exist for times $\Delta t \lesssim M^{-1}$ as allowed by the uncertainty principle. Exceptionally small temporal damping of Π^{\pm} is in general associated with sharp features in the Fourier transform $\tilde{\Pi}^{\pm}(t, \omega)$ and corresponds to

resonant phenomena. Throughout, we will not specify the interaction time τ_{int} , which has to be calculated on a case by case basis, and leave it as a free parameter.

Under the above assumptions, the first KBE (3) can be approximately solved by using the WKB method [23], which is formally an expansion of the solution to (3) in terms of the Planck constant \hbar [11]. At second order in \hbar , we find for all times t_1 and t_2 that

$$\Delta^-(t_1, t_2) = \frac{\sin\left(\int_{t_2}^{t_1} dt' \Omega_{t'}\right) e^{-\frac{1}{2}|\int_{t_2}^{t_1} dt' \Gamma_{t'}|}}{\sqrt{\Omega_{t_1}\Omega_{t_2}}}. \quad (9)$$

For small time separations, $|t_1 - t_2| \lesssim \tau_{\text{int}}$, the time-dependence of $\tilde{\Pi}^-(t, \omega)$ is negligible, and (9) can be obtained by a Laplace transformation as in [15, 32]. For $|t_1 - t_2| \gtrsim \tau_{\text{int}}$, we illustrate the derivation of the WKB result by splitting Δ^- into positive and negative frequency modes, $\Delta^- \equiv \frac{i}{2}(\Delta_{(+)}^- - \Delta_{(-)}^-)$. Inserting the functions $\Delta_{(\alpha)}^-(t_1, t_2)$ with $\alpha = \pm$ into (3) yields (for $t_1 > t_2$)

$$\omega_{\mathbf{q}}^2(t_1) - \Omega_{t_1}^2 - i\alpha\Omega_{t_1}\Gamma_{t_1} \simeq - \int_0^\infty dz \Pi^-(t_1, t_1 - z) e^{\int_{t_1}^{t_1-z} dt' (i\alpha\Omega_{t'} - \frac{1}{2}\Gamma_{t'})}, \quad (10)$$

where we neglected terms suppressed by (8) or Γ_t/Ω_t ; terms containing $\dot{\Omega}_{t_1}$ cancel out. Since $\Pi^-(t_1, t')$ effectively vanishes for $|t_1 - t'| \gtrsim \tau_{\text{int}}$ we extended the z -integration limit formally to ∞ . Ω_t and Γ_t are practically constant over the support of Π^- , hence the z -integral equals $\text{Re}\tilde{\Pi}(t_1, \hat{\Omega}_{t_1}) - i\alpha\text{Im}\tilde{\Pi}(t_1, \hat{\Omega}_{t_1})$. Then, the real and imaginary parts of (10) reproduce the definitions of Ω_t and Γ_t (7). The case $t_1 < t_2$ is treated analogously.

Knowledge of the spectral function Δ^- and the self-energies Π^\pm are enough to find a solution to the second KBE (4); it is given by the *memory integral*

$$\Delta^+(t_1, t_2) = \int_{t_i}^{t_1} dt'_1 \int_{t_i}^{t_2} dt'_2 \Delta^-(t_1, t'_1) \Pi^+(t'_1, t'_2) \Delta^-(t'_2, t_2), \quad (11)$$

where we allowed for a finite initial time t_i . This can be obtained by using the initial conditions of Δ^- and the fact that Δ^- solves the first KBE (3).

For convenience, we now split (11) into three terms,

$$\Delta^+(t_1, t_2) = \mathcal{B}(t_1, t_2) + \mathcal{C}(t_1, t_2) + \mathcal{D}(t_1, t_2). \quad (12)$$

Here, \mathcal{B} will correspond to the classical Boltzmann behavior, whereas \mathcal{C} and \mathcal{D} will give corrections that can be neglected within our approximations. The terms \mathcal{B} and \mathcal{C} are defined to contain all contributions to Δ^+ that come from the time integration over $t', t'' < t_B \equiv$

$\min(t_1, t_2)$ in (11), where we defined the *Boltzmann time* t_B . The term \mathcal{D} contains the remaining contributions that come from outside this region. For a self-energy that is exactly local in time, $\Pi^+(t_1, t_2) \sim \delta(t_1, t_2)$, only \mathcal{B} and \mathcal{C} would contribute, \mathcal{D} would be identically zero. In more general cases, \mathcal{D} is suppressed by $\Gamma_t/\tau_{\text{int}}^{-1}$ and remains thus negligible.

Now, $\mathcal{B} + \mathcal{C}$ is split such that \mathcal{B} contains all terms with equal-sign frequencies, $\mathcal{B} \sim \Delta_{(\alpha)}^- \Delta_{(\alpha)}^-$; \mathcal{C} contains the remaining opposite-sign terms, $\mathcal{C} \sim \Delta_{(\alpha)}^- \Delta_{(-\alpha)}^-$. Then, \mathcal{C} can be written as

$$\begin{aligned} \mathcal{C}(t_1, t_2) = & \frac{e^{i \int_{t_2}^{t_1} dt' \Omega_{t'}} e^{-\frac{1}{2} |\int_{t_2}^{t_1} dt' \Gamma_{t'}|}}{2\sqrt{\Omega_{t_1} \Omega_{t_2}}} \int_{-\infty}^{t_B} d\tau e^{-\int_{\tau}^{t_B} dt' (\Gamma_{t'} + 2i\Omega_{t'})} \\ & \times \int_0^{\infty} dz \frac{\Pi^+(\tau, \tau - z)}{\sqrt{\Omega_{\tau} \Omega_{\tau-z}}} e^{\int_{\tau-z}^{\tau} dt' (i\Omega_{t'} - \frac{1}{2}\Gamma_{t'})} + \text{h.c.} \end{aligned} \quad (13)$$

The first term in the second line corresponds to $\tilde{\Pi}^+(\tau, \hat{\Omega}_{\tau}^*)/\Omega_{\tau}$. Unless this expression is oscillating with frequencies $\pm 2\Omega_{\tau}$, the integral over τ is of the order $\text{Im} \tilde{\Pi}^+(\tau, \hat{\Omega}_{\tau}^*)/\Omega_{\tau}^2$ and hence negligible. Finally, \mathcal{B} can be written as

$$\begin{aligned} \mathcal{B}(t_1, t_2) = & \frac{\cos\left(\int_{t_2}^{t_1} dt' \Omega_{t'}\right) e^{-\frac{1}{2} |\int_{t_2}^{t_1} dt' \Gamma_{t'}|}}{-2\sqrt{\Omega_{t_1} \Omega_{t_2}}} \int_{-\infty}^{t_B} d\tau e^{-\int_{\tau}^{t_B} dt' \Gamma_{t'}} \\ & \times \int_0^{\infty} dz \frac{\Pi^+(\tau, \tau - z)}{\sqrt{\Omega_{\tau} \Omega_{\tau-z}}} e^{\int_{\tau-z}^{\tau} dt' (i\Omega_{t'} - \frac{1}{2}\Gamma_{t'})} + \text{h.c.} , \end{aligned} \quad (14)$$

where again we obtain $\tilde{\Pi}^+(\tau, \hat{\Omega}_{\tau}^*)/\Omega_{\tau}$ in the second line.

From (14), and adopting the common definitions

$$2\text{Re}\tilde{\Pi}^+(\hat{\Omega}_{\tau}^*) \equiv -\Omega_{\tau}(\Gamma_{\tau}^> + \Gamma_{\tau}^<) , \quad \Gamma_{\tau} \equiv \Gamma_{\tau}^> - \Gamma_{\tau}^< , \quad (15)$$

we obtain for the statistical propagator Δ^+ , up to terms that are suppressed by Γ_t/Ω_t and $\Gamma_t/\tau_{\text{int}}^{-1}$,

$$\Delta^+(t_1, t_2) \simeq \frac{\cos\left(\int_{t_2}^{t_1} dt' \Omega_{t'}\right) e^{-\frac{1}{2} |\int_{t_2}^{t_1} dt' \Gamma_{t'}|}}{2\sqrt{\Omega_{t_1} \Omega_{t_2}}} \underbrace{\int_{-\infty}^{t_B} d\tau (\Gamma_{\tau}^> + \Gamma_{\tau}^<) e^{-\int_{\tau}^{t_B} dt' (\Gamma_{t'}^> - \Gamma_{t'}^<)}}_{\equiv 1+2f(t_B)} . \quad (16)$$

We can now *define* the suggestive quantity $f(t_B)$ as function of t_B as indicated in (16). Most importantly, from its very definition it follows that $f(t_B)$ solves the BE

$$\partial_{t_B} f(t_B) = (1 + f(t_B)) \Gamma_{t_B}^< - f(t_B) \Gamma_{t_B}^> . \quad (17)$$

Then, our result for the statistical propagator becomes

$$\Delta^+(t_1, t_2) \simeq \frac{\cos\left(\int_{t_2}^{t_1} dt' \Omega_{t'}\right) e^{-\frac{1}{2}|\int_{t_2}^{t_1} dt' \Gamma_{t'}|}}{2\sqrt{\Omega_{t_1}\Omega_{t_2}}} (1 + 2f(t_B)) , \quad (18)$$

which makes explicit that under the above assumptions the memory integral (11) is governed by Boltzmann behavior. Eq. (17) can be written as $\partial_t f(t) = -\Gamma_t (f(t) - \bar{f}(t))$, with $\bar{f}(t) \equiv (\Gamma_t^>/\Gamma_t^< - 1)^{-1}$. If the χ_i are in local thermal equilibrium, the gain- and loss rates $\Gamma_t^<$ and $\Gamma_t^>$ are related by the detailed balance condition $\Gamma_t^</\Gamma_t^> = e^{-\Omega_t/T(t)}$, where $T(t)$ denotes an effective temperature; then $\bar{f}(t)$ becomes the usual Bose-Einstein distribution. Note that (17) and (18) are valid for each field mode \mathbf{q} separately.

IV. DISCUSSION

It is instructive to consider the solutions (9) and (18) locally in time, as they appear *e.g.* in loop diagrams when calculating self-energies or when deriving properties like the energy density of fields. Defining mean and relative times as $t \equiv (t_1 + t_2)/2$ and $y \equiv t_1 - t_2$, respectively, we obtain in the limit $y \ll H^{-1}$ that

$$\begin{aligned} \Delta^-(t_1, t_2) &\simeq \frac{\sin(y\Omega_t)}{\Omega_t} e^{-\frac{1}{2}|y|\Gamma_t} \quad \text{and} \\ \Delta^+(t_1, t_2) &\simeq \left[(1 + 2\bar{f}(t)) e^{-\frac{1}{2}|y|\Gamma_t} + 2\delta f(t) \right] \frac{\cos(y\Omega_t)}{2\Omega_t} , \end{aligned} \quad (19)$$

where $\delta f(t) \equiv f(t) - \bar{f}(t)$. The damping rate of the spectral function as function of y is just given by Γ_t . The damping of the statistical propagator can be understood by splitting it up in two parts as indicated (if the background is in thermal equilibrium, they correspond to the equilibrium and non-equilibrium parts of the propagator, respectively). In general, the term proportional to $\delta f(t)$ remains undamped (*cf.* discussion in [30]). However, the overall damping of Δ^+ approaches Γ_t if $|\delta f(t)| \ll 1$. In this case, and locally in time, our results reproduce the common Kadanoff-Baym ansatz in the weak damping limit (see *e.g.* Refs. [4, 6, 7, 29–31]). This implies that $f(t)$ as defined in (16) indeed plays the role of a generalized phase space distribution function of effective plasma excitations. Note that Γ_t^{\geq} may include off-shell transport [10, 15, 32, 33].

Using (11), we can discuss the effect of boundary conditions at finite time ($t_i = 0$ for definiteness). If Π^+ is nonsingular at $t_1 = t_2 = 0$, (11) implies the initial conditions $\Delta_i^+ =$

$\dot{\Delta}_i^+ = \ddot{\Delta}_i^+ = 0$; here, we defined $\Delta_i^+ \equiv \Delta^+|_{t_1=t_2=0}$, $\dot{\Delta}_i^+ \equiv \partial_{t_1}\Delta^+|_{t_1=t_2=0} = \partial_{t_2}\Delta^+|_{t_1=t_2=0}$ and $\ddot{\Delta}_i^+ \equiv \partial_{t_1}\partial_{t_2}\Delta^+|_{t_1=t_2=0}$. Arbitrary initial conditions for Δ^+ can be implemented by formally adding $\delta\Pi^+(t_1, t_2) = -\partial_{t_1}\partial_{t_2}\Delta^+(t_1, t_2)\delta(t_1)\delta(t_2)$ to $\Pi^+(t_1, t_2)$ in (11). Boltzmann behavior according to (18) only arises when $\dot{\Delta}_i^+ = 0$ and $\Omega_t\Delta_i^+ = \Omega_t^{-1}\ddot{\Delta}_i^+ = \frac{1}{2} + f|_{t=0}$; otherwise, (11) generates oscillating terms that are exponentially damped away with the rate Γ_t .

In the early universe, quantum fields propagate in an expanding background described by a Friedmann-Robertson-Walker metric with scale factor $a(t)$ and Hubble rate $H \equiv \dot{a}/a$. Since in conformal coordinates this is equivalent to a time-dependent mass term, expansion is already accounted for in our calculation. The BE (17) remains unchanged, but the distribution function becomes a function of comoving momentum $\mathbf{q}_{\text{com}} = a(t)\mathbf{q}$. Going back to units of physical space and momentum yields the well-known result

$$\partial_t f_{\mathbf{q}}(t) = (1 + f_{\mathbf{q}}(t))\Gamma_t^< - f_{\mathbf{q}}(t)\Gamma_t^> + H\mathbf{q}\nabla_{\mathbf{q}}f_{\mathbf{q}}(t). \quad (20)$$

V. CONCLUSIONS

We have presented a simple method to derive BEs from the fundamental KBEs, and illustrated it in case of a single real scalar field ϕ . Instead of performing a gradient expansion on the KBEs, we derived approximate WKB solutions to the full KBEs that are valid under the physical assumptions of weak coupling and separation of interaction and scattering time scales. As a result, we obtain analytical expressions for the two-point functions of ϕ —given by the spectral function $\Delta^-(t_1, t_2)$ and the statistical propagator $\Delta^+(t_1, t_2)$ —that are valid at all times. Locally in time, our results reproduce the common Kadanoff-Baym ansatz. Although we formally neglected backreaction, it can in principle be included by reinserting our explicit solutions for Δ^\pm into the calculation of the ϕ self-energy; if the self-energies remain well-behaved, a self-consistent description in terms of BEs emerges. In any case, as long as (WKB-)solutions for the spectral functions can be found, the presented approach can be extended to multi-flavor problems, flavor oscillation and to fermionic fields [34], as well as beyond the weak damping regime. No assumptions about the structure of distribution functions in Wigner-space needs to be made. Instead, the validity of a description in terms of distribution functions can be directly read off from a comparison between spectral functions and statistical propagators.

ACKNOWLEDGMENTS

We are very grateful to Jürgen Berges, Mathias Garny and Georg Raffelt for useful discussions and comments on the manuscript.

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